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Unitary Irreducible Representations of $SL(n, \mathbb{C}) \cdot \mathbb{R}(n^2)$. I

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The family of Lie groups $SL(n, \mathbb{C}) \cdot \mathbb{R}(n^2)$ is studied from the mathematical point of view, i.e., their structure and unitary irreducible representations. The first steps in the study of representations are based on the Wigner-Mackey-Bruhat method. The homogeneous spaces and little groups are determined by introducing a (generally) pseudo-Hermitian metric on vector spaces of finite dimension. However, the presence of null vectors does not allow this approach to be an exhaustive one; there are cases where the stabilizer is a semidirect product of a reductive group by a nilpotent one. In such cases, results are available if the reductive subgroup is compact, but it seems that these results can be generalized. The connection is also examined, both in structure and in representations, between this family and creation-annihilation operators. A complete list of inequivalent classes of representations is given in terms of orbits and little groups for $n = 2, 3, 4, 6$.

INTRODUCTION

Since it was proved that compact Lie groups are an incomplete tool for treating the strong-interaction symmetries, several other types of groups have been proposed. Although most of them are semisimple groups, "inhomogeneous" groups have also been proposed.¹⁻⁷ Semisimple groups have the advantage of being a convenient tool for which there is an abundant mathematical literature. However, in spite of several hints, we do not know yet the symmetry group; nonsemisimple groups like the inhomogeneous special linear groups, especially those which contain the Poincaré group, may prove to be of great interest, and they appear as possible solutions to the external-internal unification problem. Thus, it is necessary to investigate their structure and their unitary irreducible representations in order to identify quantum numbers. Besides, there are other physical domains where such groups may be useful, for example, the classification of nuclear spectra and the creation-annihilation operators. The physical applications of this family will be treated in a subsequent article.

Here, we are interested in the mathematical study of these groups, i.e., the investigation of their structure and unitary irreducible representations (UIR's). The problem was solved by Wigner⁸ for the Poincaré group ($n = 2$). Then Mackey^{9,10} and Bruhat¹¹ proved that the method of induced representations can be applied,

with some topological restrictions, to any group containing a normal Abelian subgroup. Thus, when dealing with such groups, one needs to separate a vector space into orbits and to find the little groups that leave unchanged a point of each orbit, the choice of this point being arbitrary.

In the case of $G_n = SL(n, \mathbb{C}) \cdot \mathbb{R}(n^2)$,¹² one obtains either reductive little groups or semidirect products of reductive groups by Abelian groups.¹³ Explicit results are available for the representations of the first case.¹⁴ The second case is more interesting: Further Wigner decomposition can generally be done only once more, because the next little groups' structure is more complicated (nilpotent subgroups take the place of Abelian ones). The investigation is carried on in some cases where the reductive subgroup is compact, with other methods consisting of the decomposition of the regular representation and the establishment of a complete correspondence between the representations of the reductive subgroup and the faithful representations of the whole semidirect product which occur in the regular representation.

On the other hand, a connection is established between Lie algebras of the family and creation-annihilation operators by means of the symplectic Lie algebra, which is identified to a subalgebra of the universal enveloping algebra generated by these operators. Any algebra of the family is isomorphic to a subalgebra



of some symplectic algebra: One can thus establish both structural and spectral connections among infinitesimal operators. The application of these results to field theory will be treated in a subsequent article.

Section I deals with the group structure and *TCP* generalization; the action of these operators is studied both in abstract group structure and in representations. The study of UIR's of proper G_n begins in Sec. II, where Wigner's method is used in two successive steps. In Sec. III, special cases of decomposition are examined, where Wigner's method can be applied. Section IV deals with cases in which difficulties due to nilpotent subgroups occur. Creation-annihilation operators are treated in Sec. V.

It is important to note the following: To avoid confusion, we specify that, when we speak of representations of a Lie group, we mean UIR's unless the contrary is specified; the description of representations is given up to those of reductive little groups, such as $U(p, q)$ or $GL(r, \mathbb{C})$, or their direct products.¹⁴

I. GENERALITIES (TRIVIAL STRUCTURAL REMARKS)

A. G_n and Poincaré Subgroups

The group G_n is the semidirect product of the unimodular complex linear group $SL(n, \mathbb{C})$ by the Abelian group $R(n^2)$ of n^2 real translations. The action of $SL(n, \mathbb{C})$ on $R(n^2)$ is

$$(A)_{\alpha\beta} \rightarrow \Lambda_{\alpha}^{\gamma} \bar{\Lambda}_{\beta}^{\delta} (A)_{\gamma\delta}, \quad \Lambda \in SL(n, \mathbb{C}), \quad A \in R(n^2).$$

In matrix notation, the generic element of G_n can be written

$$g = \begin{pmatrix} \Lambda & A\Lambda^{*-1} \\ 0 & \Lambda^{*-1} \end{pmatrix},$$

where Λ and A are $n \times n$ complex matrices, Λ is unimodular, and A is Hermitian (or A is skew-Hermitian). The group law is

$$(A, \Lambda)(A', \Lambda') = (A + \Lambda A' \Lambda^*, \Lambda \Lambda').$$

There are many ways to choose a subgroup of G_n isomorphic to the universal covering of the inhomogeneous Lorentz group $\mathcal{P}$. However, we shall immediately put in evidence the most interesting choice in the case $n = 2n'$. In fact, $R(4n^2) = R(4) \times R(n'^2)$, that is, every Hermitian matrix of $2n'$ lines and $2n'$ columns can be written as a finite sum of the generators $\sigma_{\mu} \times \alpha_i$, where α_i is an $n' \times n'$ Hermitian matrix and σ_{μ} is an ordinary Pauli matrix. Similarly, the matrices $\lambda \times \lambda'$ [where $\lambda \in \mathfrak{gl}(2, \mathbb{R})$, $\lambda' \in \mathfrak{gl}(n', \mathbb{R})$ and such that $Sp\lambda \cdot Sp\lambda' = 0$] span $\mathfrak{sl}(2n', \mathbb{R})$. The complexification of this algebra gives $\mathfrak{sl}(2n', \mathbb{C})$ and, if one puts the equivalence relations

$$(i\lambda) \times (\lambda') = (\lambda) \times (i\lambda') = i(\lambda \times \lambda'),$$

$$(i\lambda) \times (i\lambda') = -\lambda \times \lambda',$$

one finds again the commutation relations of $\mathfrak{g}_{2n}$, by using those of $\mathfrak{g}_2$ and $\mathfrak{g}_{n'}$. The Poincaré subalgebra is generated by $\{a_i \times 1, b_i \times 1; \sigma_{\mu} \times 1\}$, and its commutator in $\mathfrak{gh}(2n', \mathbb{C})$ is the subalgebra isomorphic to $\mathfrak{su}(n')$, spanned by $\{1 \times u\}$. (Here a_i and b_i are generators of the Lorentz Lie algebra.)

B. Involutive External Automorphisms of G_n

In view of further physical applications, it is important to investigate whether our group G_n can be imbedded into a larger group $\tilde{G}_n$ containing "parity," that is, whether $\tilde{G}$ is an extension of Z_2 by G , since we require the operator parity to be an involution and to correspond to the physical parity for the Poincaré group ($n = 2$), or a particular Poincaré subgroup of G_n . For this purpose, we use the method introduced in Ref. 6; $\tilde{G}$ is a semidirect product of $Z_2 = \{1, s\}$ by G . The operator parity is $P = \text{Ad } s$ and it induces an automorphism on the Lie algebra $\mathfrak{g}$. Therefore, we investigate the different possibilities for $\text{Ad } s \in \text{Aut } \mathfrak{g}$. In order to determine $\text{Aut } \mathfrak{g}$ (by a method similar to that used in Ref. 15), we first remark that $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{t}$ (considered as vector spaces), with $\mathfrak{f} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{t} = R(n^2)$, $\mathfrak{t}$ being also a characteristic ideal of $\mathfrak{g}$. We can thus write, for every $F \in \text{Aut } \mathfrak{g}$,¹⁶

$$F = (f, \Phi, \psi),$$

with

$$f \in \text{Aut } \mathfrak{f} = GL(n^2, \mathbb{R}),$$

$$\Phi \in \text{Aut } \mathfrak{f} = (SL(n, \mathbb{C})/Z_n) \cdot (Z_2 \times Z_2)$$

$$= \text{Int } \mathfrak{f} \cdot (Z_2 \times Z_2),$$

$$\psi \in \mathfrak{L}(\mathfrak{f}; \mathfrak{t}) = \mathfrak{L}(R(2n^2 - 2); R(n^2)).$$

The group law establishes a relation between Φ and f : Let $p \in \text{Hom}(\mathfrak{sl}(n, \mathbb{C}); \mathfrak{gl}(n^2, \mathbb{R}))$ be such that $p(K) \cdot T = KT + TK^*$; f must belong to the normalizer N of $p(\mathfrak{f})$ in $GL(n^2, \mathbb{R})$, because

$$f \cdot p(K) \cdot f^{-1} = p(\Phi(K));$$

thus, f defines Φ as an automorphism of $\mathfrak{f}$. Calculating that, if $n > 2$, the external automorphism $K \rightarrow -K^*$ cannot be expressed by any $\text{Ad } f \in N$, we easily check that N is the direct product of $\text{Aut } \mathfrak{f}/Z_2$ by the multiplicative group of real numbers.

Using the group law again, one can prove that

$$\psi(S) = p(\Phi(S)) \cdot A = \Phi(S)A + A[\Phi(S)]^*,$$

where A is a fixed $n \times n$ matrix. One can thus write $\text{Aut } \mathfrak{g} = (R^* \times \text{Aut } \mathfrak{f}/Z_2) \cdot R(n^2) = (R^* \times Z_2) \cdot \text{Int } \mathfrak{g}$ and

$$\begin{aligned} F_{\lambda, j, \Lambda, A}(K; T) &= (\Lambda \cdot jK \cdot \Lambda^{-1}; \lambda\Lambda \cdot jT \cdot \Lambda^* \\ &\quad + p(\Lambda \cdot jK \cdot \Lambda^{-1}) \cdot A) \\ &= \text{Ad } (-A, \Lambda) \cdot (jK; \lambda jT), \end{aligned}$$

j being either 1 or J , with $JK = \bar{K}$ and $JT = \bar{T}$.



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Searching for an involutive automorphism F , we see that λ must be equal to ± 1 and that there always exists an A' such that

$$F_{\pm 1, j, \Lambda, A} = F_{\pm 1, j, \Lambda, 0} \cdot F_{1, 1, 1, A'}.$$

Thus, we can only investigate semisimple automorphisms, which leave globally invariant $\mathfrak{t}$ and $\mathfrak{f} = \text{Ad } A'(\mathfrak{l}) \cong \mathfrak{sl}(n, \mathbb{C})$.

We now consider the case $j = 1$. Then $\text{Ad } \Lambda^2 = I$ and $\Lambda^2 = \exp(2ik\pi/n) \cdot I$. Λ is semisimple and may always be put into the diagonal form $I_p \oplus (-I_{2q})$ or $\exp(in/n) \cdot [I_p \oplus (-I_{2q+1})]$. Then λ must be -1 , since we want P to be external. The positive and negative eigenspaces of P on the semisimple part both have a complex structure, $\mathfrak{f}_+$ being a subalgebra [see (A) below].

In the case $j = J$, we have $\text{Ad } (\Lambda\bar{\Lambda}) = I$, and λ may be either 1 or -1 ; the choice of λ affects only the eigenvalues of the eigenvectors of P belonging to $\mathfrak{t}$, while P defines in $\mathfrak{f}$ an external real form¹⁷ $\mathfrak{f}_+$ —[see (B) and (C)].

We now write the results in terms of positive and negative eigenspaces of P :

$$\begin{aligned} \text{(A)} \quad \mathfrak{f}_+ &= \mathfrak{sl}(p, \mathbb{C}) + \mathfrak{gl}(q, \mathbb{C}), & \mathfrak{f}_- &= \mathbb{C}(p \times q) + \mathbb{C}(q \times p), \\ \mathfrak{t}_+ &= \mathbb{C}(p \times q), & \mathfrak{t}_- &= \mathbb{R}(p^2) + \mathbb{R}(q^2), \\ \text{(B)} \quad \mathfrak{f}_+ &= \mathfrak{sl}(n, \mathbb{R}), & \mathfrak{t}_- &= i \cdot \mathfrak{sl}(n, \mathbb{R}), \\ \mathfrak{t}_\pm &= \mathbb{R}(\tfrac{1}{2}n(n+1)), & \mathfrak{t}_\pm &= \mathbb{R}(\tfrac{1}{2}n(n-1)), \\ \text{(C)} \quad \mathfrak{f}_+ &= \mathfrak{su}^*(2n'), & \mathfrak{f}_- &= i \cdot \mathfrak{su}^*(2n'), \quad 2n' = n, \\ \mathfrak{t}_\pm &= \left\{ \begin{pmatrix} H & -A \\ A & \bar{H} \end{pmatrix} \right\}, & \mathfrak{t}_\mp &= \left\{ \begin{pmatrix} H & S \\ S & -\bar{H} \end{pmatrix} \right\}, \end{aligned}$$

where H , A , and S are $n' \times n'$ complex matrices, with H Hermitian, A skew-symmetric, and S symmetric.

C. Generalized PCT

Having defined P , we may as well define (external) operators T and C , in order to generalize the PCT assignment. We impose $PCT = e^{i\varphi}$, C and T both being of square unity and being represented by an anti-unitary operator whenever P is represented by a unitary operator. Evidently, we need to define only one of them, say T , C being then defined by $C = PTe^{i\varphi}$. Following again the method of Flato and Sternheimer,⁶ we have two choices. In the first one, we take T to be the principal anti-automorphism, which turns $\mathfrak{g}$ to its opposite Lie algebra, having again the possibility of combining it with the involution which has value $+1$ on $\mathfrak{f}$ and -1 on $\mathfrak{t}$. The second choice consists of defining T , not by its action on the algebra's structure, but on the representations of the group $\tilde{G}$. We consider two complex conjugate

representations U and $\bar{U}$ of $\tilde{G}$: T is then defined by

$$U(Tg) \cdot \varphi = \bar{U}(g) \cdot \bar{\varphi}, \quad g \in \tilde{G}, \text{ for every vector } \varphi.$$

Both definitions give an antiunitary operator which is not an element of $\tilde{G}$. We have thus given a formal extension of the TCP assignment to the family of groups G_n which contain the inhomogeneous Lorentz group.

D. Representations of $\tilde{G}$

To determine representations of $\tilde{G}$ knowing those of G , let $\check{g} = sgs$. We can thus associate couples of representations of G ; that is, with U we associate the representation $\check{U}$, $\check{U}(g) = U(sgs)$. The representations U and $\check{U}$ are, in general, inequivalent. This always happens when U is unitary irreducible and not reduced to unity on the Abelian subgroup. In this case, we proceed as follows: We "double" the Hilbert space of U , and we put

$$\begin{aligned} U(g) &= \begin{pmatrix} U(g) & 0 \\ 0 & U(sgs) \end{pmatrix}, & \text{if } g \in G, \\ &= \begin{pmatrix} 0 & U(g) \\ U(sg) & 0 \end{pmatrix}, & \text{if } f \in sG. \end{aligned}$$

Thus, we obtain UIR's of G in which the discrete element s is represented by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These representations provide a useful tool for the definition of P , C , and T on known representation spaces of G .

II. ORBITS AND STABILIZERS

A. First Decomposition

The method used here to determine representations of G is that used by Wigner⁸ for the Poincaré group. Mackey^{9,10} and Bruhat¹¹ have proved since then that this method can be generalized to any semidirect product of a reductive Lie group by an Abelian one. We thus have to decompose the dual to the Abelian subgroup into orbits and then find the representations of the stabilizer of one point of each orbit.

In case of G_n , the dual $\hat{\mathbb{R}}(n^2)$ is isomorphic to $\mathbb{R}(n^2)$; if both are represented by $n \times n$ Hermitian matrices, the bilinear form is

$$\langle \alpha, A \rangle = \text{Sp} \alpha A, \quad \alpha \in \hat{\mathbb{R}}(n^2), \quad A \in \mathbb{R}(n^2). \quad (1)$$

$SL(n, \mathbb{C})$ acts on $\hat{\mathbb{R}}(n^2)$ according to

$$\alpha \cdot \Lambda = \Lambda^* \alpha \Lambda, \quad \Lambda \in SL(n, \mathbb{C}). \quad (2)$$

It is then easy to verify that an orbit is determined by $|\det \alpha| = M^n$ and the triplet $(p, q, r) = \text{sign } \alpha$, the three integers denoting, respectively, the number of positive, negative, and zero eigenvalues of α .

Taking for standard element in each orbit the matrix

$$(I_p \oplus (-I_q) \oplus 0_r), \text{ if } M = 0,$$

and

$$M \cdot (I_p \oplus (-I_q)), \text{ if } M \neq 0,$$

we have the following result.

Proposition 1: The little group corresponding to the orbit $\{M; (p, q, r)\}$ depends only on (p, q, r) and is

$$S(U(p, q) \times GL(r, \mathbb{C})) \cdot C(r(p + q)).$$

Remark: By $S(G^k \times G'^{k'})$, when speaking of matrix groups, we mean the subgroup of $G \times G'$ determined by $(\det g)^k \cdot (\det g')^{k'} = 1$.

We immediately obtain the following corollaries.

Corollary 1: Every nondegenerate representation of G_n is determined by a positive number M , a partition of n into two integers (p, q) , and a representation of $SU(p, q)$.

Corollary 2: Most degenerate representations of G_n are all the representations of $SL(n, \mathbb{C})$.

B. Second Decomposition

When $r(p + q) \neq 0$, the little group G' is not semisimple but is still a semidirect product of a reductive by an Abelian group. So we still apply the same method to determine its representations.

G' is a matrix group: any $g' \in G'$ can be written

$$g' = \begin{pmatrix} u & 0 \\ zu & \mu \end{pmatrix}, \quad u \in U(p, q), \quad \mu \in GL(r, \mathbb{C}), \\ z \in \mathbb{C}^{p+q}, \quad \det u \cdot \det \mu = 1.$$

Taking the dual Abelian group isomorphic to be $\mathbb{C}(r; \mathbb{C}^{p+q})$, we have the bilinear form

$$\langle \zeta, z \rangle = \frac{1}{2} Sp(z\zeta + \zeta^* z^*). \quad (3)$$

The action of the homogeneous part on the dual space is

$$\zeta \xrightarrow{g'} u^{-1} \zeta \mu.$$

To decompose $\mathbb{C}(p + q; \mathbb{C}(r))$ into orbits of G' , we observe that $u^{-1} \zeta \mu$ denotes the same linear mapping as ζ , after changes of bases in $\mathbb{C}(r)$ and $\mathbb{C}(p + q)$,

the latter being a (p, q) -pseudo-unitary one. Rank ζ is an invariant. Moreover, there is a metric induced on $\text{Im } \zeta$ by the (p, q) metric of $\mathbb{C}(p + q)$: There exists a basis of $p_1 + q_1 + s_1$ mutually orthogonal vectors in $\text{Im } \zeta$, p_1 of them having a positive square, q_1 a negative one, and s_1 a null one. The triplet (p_1, q_1, s_1) is an invariant.

Remark: The following inequalities obviously hold:

$$\text{Rank } \zeta = p_1 + q_1 + s_1 \leq \min(p + q, r),$$

$$p_1 + s_1 \leq p,$$

$$s_1 \leq \min(p, q),$$

$$q_1 + s_1 \leq q.$$

Furthermore, if $\text{rank } \zeta = r \leq p + q$, we may associate with every ζ a vector ζ_0 , the components of which are the minors of rank r of ζ . μ acts on ζ_0 like a scalar of modulus one, and u^{-1} acts according to the r th fundamental (nonunitary) representation of $SU(p, q)$,¹⁸ both keeping invariant a pseudo-Hermitian product in $\mathbb{C}(\binom{p+q}{r})$. The metric tensor is then

$$g(i_1, i_2, \dots, i_r; j_1, j_2, \dots, j_r) \\ = g(i_1; j_1) \cdot g(i_2; j_2) \cdots g(i_r; j_r),$$

with

$$i_1 < i_2 < \cdots < i_r.$$

There are no other invariants characterizing the orbits. We can thus state a theorem.

Theorem 1: Orbits of $\mathbb{C}(r; \mathbb{C}(p + q))$ are characterized by $\text{sign } \zeta = (p_1, q_1, s_1)$ and the real positive number $M_0 = |\langle \zeta_0 | \zeta_0 \rangle|$. M_0 is always zero if $\text{rank } \zeta < r$.

Remark 1: If $p + q = r$, $M_0 = |\det \zeta|^2$.

Remark 2: It is easy to check that orbits which differ only on M_0 have isomorphic stabilizers.

To determine the little group, we first find out its Lie algebra. Putting $u = \exp U$ and $\mu = \exp M$, we find

V_1	$-E^*$	$+E^*$	0
EI_1	$\frac{1}{2}(\Lambda_1 - \Lambda_1^*) + iH$	$\frac{1}{2}(\Lambda_1 + \Lambda_1^*) - iH$	FI_1'
EI_1	$\frac{1}{2}(\Lambda_1 + \Lambda_1^*) + iH$	$\frac{1}{2}(\Lambda_1 - \Lambda_1^*) - iH$	FI_1'
0	$-F^*$	F^*	V_1'

$$M = \begin{bmatrix} V_1 & 0 & 0 \\ E & \Lambda_1 & 0 \\ X & Y & M_1 \end{bmatrix}$$

with

$$V_1 \in \mathfrak{su}(p_1, q_1), \quad E \in \mathbb{C}((p_1 + q_1)s_1),$$

$$V'_1 \in \mathfrak{su}(p'_1, q'_1), \quad F \in \mathbb{C}((p'_1 + q'_1)s_1),$$

$$\Lambda_1 \in \mathfrak{gl}(s_1, \mathbb{C}), \quad X \in \mathbb{C}((p_1 + q_1)r_1),$$

$$M_1 \in \mathfrak{gl}(r_1, \mathbb{C}), \quad Y \in \mathbb{C}(s_1 \cdot r_1),$$

$$H \in \mathbb{R}(s_1^2) \text{ (Hermitian),}$$

$$I_1 = I(p_1) \oplus (-I(q_1)), \quad I'_1 = I(p'_1) \oplus (-I(q'_1)),$$

$$p'_1 = p - p_1 - s_1, \quad q'_1 = q - q_1 - s_1,$$

$$r_1 = r - p_1 - s_1 - q_1,$$

$$2SpV_1 + SpV'_1 + Sp(2\Lambda_1 - \Lambda_1^*) + SpM_1 = 0.$$

On the other hand, it is easy to find the discrete center of the little group; it is $\mathbb{Z}_n$.

Theorem 2: The stabilizer G'' of the orbit (p_1, q_1, s_1) of G' is the inessential extension of a reductive group R by a nilpotent one N ; R is the group

$$S(U^{p_1, q_1}) \times U(p'_1, q'_1) \times GL^{2;1}(s_1; \mathbb{C}) \times GL(r_1; \mathbb{C});$$

N is a central extension of the Abelian group A by the Abelian group B such that

$$A = \mathbb{C}(\mathbb{C}(p_1 + q_1), \mathbb{C}(s_1))$$

$$\otimes \mathbb{C}(\mathbb{C}(p'_1 + q'_1), \mathbb{C}(s_1)) \otimes \mathbb{C}(\mathbb{C}(s_1), \mathbb{C}(r_1)),$$

$$B = \mathbb{C}(\mathbb{C}(p_1 + q_1), \mathbb{C}(r_1)) \otimes \mathbb{R}(s_1^2).$$

The group law in N is

$$\begin{aligned} n \cdot n' &= (H, X, Y, E, F)(H', X', E', F') \\ &= (H + H' + \frac{1}{2}(EI_1E'^* - E'I_1E^*) \\ &\quad + \frac{1}{2}(FI_1F'^* - F'I_1F^*), X + X' \\ &\quad + \frac{1}{2}(YE' - Y'E), Y + Y', E + E', F + F'). \end{aligned}$$

The law in G'' , expressed as a semidirect product, is

$$\begin{aligned} (v_1, v'_1, \lambda_1, \mu_1) \cdot (H, X, Y, E, F) \\ = (\lambda_1 H \lambda_1^*, \mu_1 X v_1^{-1}, \mu_1 Y \lambda_1^{-1}, \lambda_1 E v_1^{-1}, \lambda_1 F v_1^{-1}) \\ \cdot (v_1, v'_1, \lambda_1, \mu_1). \end{aligned}$$

Remark: The exponent $2;1$ means that, in the determinant condition, one must take $\det(\lambda^2 \lambda^{*-1})$.

C. Third Decomposition

To study representations of group G'' is much more tedious (in the general case) than in previous decompositions, because the normal subgroup of the semidirect product is no longer Abelian but, generally, nilpotent. The extension structure of G'' is expressed in

this diagram:

$$\begin{array}{ccccccc} & 1 & & 1 & & 1 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 & \longrightarrow & B & \longrightarrow & N & \longrightarrow & A \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & B & \longrightarrow & G'' & \longrightarrow & G''/B \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & \longrightarrow & R & \longrightarrow & R \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array}$$

The maximal Abelian normal subgroup is, in general, $B \otimes \mathbb{C}(s_1; \mathbb{C}(r_1))$; the action of G'' on the dual is

$$\begin{pmatrix} \hat{H} \\ \hat{X} \\ \hat{Y} \end{pmatrix} \rightarrow \begin{pmatrix} \lambda_1^* \hat{H} \lambda_1 \\ u_1^{-1} \hat{X} \mu_1 \\ \lambda_1^{-1} (\hat{Y} - E \cdot \hat{X}) \mu_1 \end{pmatrix}.$$

Orbits of $\hat{H}$ and $\hat{X}$ have been calculated in Secs. II.A and II.B. There is, however, a difference: when $\text{rank } \hat{H} = s_1$ and $\text{rank } \hat{X} = r_1$, no determinants nor pseudonorms are conserved, since λ_1 and μ_1 are not unimodular matrices but only obey $|\det \lambda_1 \cdot \det \mu_1| = 1$. Thus, we have only the following invariant quantity:

$$\|\hat{H}\| \times \|\hat{X}\|^2,$$

where $\|\hat{H}\| = |\det \hat{H}|$ and $\|\hat{X}\|$ is the $(p_1^{+q_1})$ pseudo-norm of $\hat{X}$ as defined in (2).

Orbits of $\hat{Y}$ depend on those of $\hat{X}$. Let $\mathbb{C}(r_1) = E_1 \oplus E_0$ with $E_0 = \text{Ker } \hat{X}$; then $\hat{Y}$ is the sum of its restrictions to E_1 and E_0 , i.e., $\hat{Y} = \Psi_1 + \Psi_0$. Then it is easy to see that $\text{rank } \Psi_0$ is the only discrete invariant characterizing orbits of $\hat{Y}$. There is no continuous invariant if $\text{rank } \hat{X} + \text{rank } \Psi_0 < r_1$. However, such invariants may arise in special cases, as, for instance, when $\hat{X} = 0$ and $s_1 = r_1$, where $\|\hat{H}\hat{Y}\|$ is conserved.

The stabilizers will be of the form $G''' = R' \cdot (AB)$, where R' is a reductive group, determined as before. But here appears the following fact: In the extension, the maximal Abelian subgroup is no longer the second factor of a semidirect product, since its complementary is not a subgroup of G''' . Thus, one cannot consider R' by itself nor $R' \cdot B = G'''/A$ and apply again the Mackey-Bruhat procedure. If one tries to apply it to the whole group G''' , one finds the same maximal Abelian normal subgroup, the orbits being points and the stabilizer being G''' itself. So far, no method has been devised that gives all representations of such groups.

We must note, however, that these difficulties arise when n becomes large. Few cases occur for $n \leq 6$, and the decomposition problems for larger n seem to present no physical interest. So we consider some

special cases in the next section, trying to cover all cases for small n .

III. SPECIAL CASES OF DECOMPOSITION

In this section, we consider some cases where further decomposition may be carried on; i.e., N is Abelian.

A. Orbits without Null Vectors

Since difficulties arise for the orbits of Sec. II.B which contain null vectors but not their positive and negative components, we shall consider orbits where such vectors do not occur, i.e., $s_1 = 0$. Then G'' is the group

$$\underline{N}(\underline{U}(p_1, q_1) \times U(p'_1, q'_1) \times \underline{GL}(r_1)) \cdot C(r_1(p_1 + q_1)),$$

where only the underlined subgroups act on

$$C(r_1(p_1 + q_1))$$

(and where p'_1 may be smaller than q'_1). The decomposition is already given by Sec. II.B and (r_1, p_1, q_1) is decomposed into $p_2, q_2, p'_2, q'_2, s_2, r_2$. In the same way, (r, p, q) gives rise to $p_1, q_1, p'_1, q'_1, s_1, r_1$. If we take only orbits for which $s_2 = 0$, for the same reasons as above, and repeat this procedure until we arrive at a reductive stabilizer, we obtain the series

$$s(U(p'_1, q'_1) \times U^2(p'_2, q'_2) \times \cdots \times U^k(p'_k, q'_k) \times U^{k+1}(p_k, q_k) \times GL(r_k)),$$

such that

$$n = r_k + \sum j \cdot (p'_j + q'_j) + (k+1) \cdot (p_k + q_k).$$

We distinguish two series inside this formula: series (1) if $r_k \neq 0$, and series (2) if $r_k = 0$.

A stabilizer belonging to (1) determines only one orbit, since this notation makes a distinction between $U(p'_j, q'_j)$ and $U(q'_j, p'_j)$ for $p'_j \neq q'_j$, and thus all successive ranks and signatures are given. On the other hand, to the stabilizers belonging to (2) there corresponds a continuous spectrum of orbits since the $\binom{p_k+q_k}{p_{k-1}+q_{k-1}}$ pseudonorm varies throughout all strictly positive real values and does not affect the stabilizer. So we have:

Theorem 3: There are two series of UIR's of G . Representations of series (1) are in 1-to-1 correspondence with those of the little group (1). Representations of the second series are determined by those of the little group (2) and by a strictly positive number.

Remark: p'_j and q'_j may be zero, but $p_k + q_k \neq 0$ (respectively, $r_k \neq 0$) in case (2) [respectively (1)]. k may take any value from 0 to $n-1$ (provided we put $p_0 = p, q_0 = q, r_0 = r, p'_0 = q'_0 = 0$).

B. Orbits with Null Vectors

Another case where the difficulties due to nilpotency drop out is when $p_1 = q'_1 = q_1 = q' = 0$ (then $p = q = s \leq r$); there are different results for $s_1 = r_1$ and $s_1 < r_1$.

Case A: $s_1 = r_1$: The group G'' is then

$$Z_3 \times SL(s_1; \mathbb{C}) \cdot R(s_1^2),$$

since the determinant condition gives

$$\det \lambda_1 = \exp(\frac{2}{3}ik\pi).$$

We have immediately:

Theorem 4: If $n = 3n'$, to every representation of G'_n there correspond three representations of G_n .

Remark: The same type of little group may occur as a direct factor after the same steps of decomposition described in Sec. III.A are carried out, the other direct factor being reductive.

Case B: $s_1 < r_1$: Let $(\Delta, \Psi) = (\hat{H}, \hat{Y})$ be an element of the dual Abelian subgroup. It transforms by

$$\begin{pmatrix} \Delta \\ \Psi \end{pmatrix} \rightarrow \begin{pmatrix} \lambda_1^* \Delta \lambda_1 \\ \lambda_1^{-1} \Psi \mu_1 \end{pmatrix}.$$

Rank Δ , sign Δ , and rank Ψ are discrete invariants, but, since Δ and Ψ do not vary independently, orbits cannot yet be characterized. So we consider the orbit of Ψ when Δ is a fixed (p', q', r') Hermitian form. As a result, λ_1 must be of the form

$$\lambda_1 = P_0 \lambda_1 P_0 + P_0 \lambda_1 P_1 + P_1 \lambda_1 P_1,$$

where $P_1 \lambda_1 P_1$ belongs to a $U(p', q')$ subgroup [in particular, $|\det(P_1 \lambda_1 P_1)| = 1$] and P_0 and P_1 are Hermitian idempotents such that

$$P_0 + P_1 = I, \quad P_0 \Delta P_0 = 0, \quad P_1 \Delta P_1 = \Delta.$$

Since $\Psi = P_0 \Psi + P_1 \Psi$, let us consider the orbit of $P_1 \Psi$ first:

$$\begin{aligned} P_0 \Psi &\rightarrow (P_0 \lambda_1^{-1} P_0) \cdot (P_0 \Psi) \cdot \mu_1 + (P_0 \lambda_1^{-1} P_1) \cdot (P_1 \Psi) \cdot \mu_1, \\ P_1 \Psi &\rightarrow (P_1 \lambda_1^{-1} P_1) \cdot (P_1 \Psi) \cdot \mu_1. \end{aligned}$$

According to previous results (see Sec. II.B), rank and signature of the vector space spanned by the column vectors of $P_1 \Psi$ are invariant. Calculating the corresponding little group by introducing projectors Q_0 and Q_1 , such that $Q_0 + Q_1 = I$, $P_1 \Psi Q_1 = P_1 \Psi$, and $P_1 \Psi Q_0 = 0$, one obtains, in particular, $Q_1 \mu_1 Q_0 = 0$.

and, thus,

$$P_0 \Psi Q_0 \rightarrow (P_0 \lambda_1^{-1} P_0) \cdot (P_0 \Psi Q_0) \cdot (Q_0 \mu_1 Q_0)$$

and rank $P_0 \Psi Q_0$ is a constant, while $P_0 \Psi Q_1$ varies arbitrarily.

Searching for continuous spectra of invariants, we observe that $|\det(P_0 \lambda_1^{-1} P_0)| = |\det(Q_0 \mu_1 Q_0)| > 0$; so there is no continuous spectrum related to $(P_0 \Psi Q_0)$. Examining $P_1 \Psi$, we see that $|\det \mu_1|$ is arbitrary, unless $P_1 = 1$ (i.e., rank $\Delta = s_1$), in which case

$$|\det \mu_1| = \det |(P_1 \lambda_1 P_1)| = 1.$$

Then there exists a pseudonorm for $P_1 \Psi = \Psi_1$, if rank $\Psi = \text{rank } \mu_1 = r_1$. Finally, no continuous spectrum exists for Δ . Thus we have the following result.

- Proposition:* Orbits are characterized by sign Δ and
- (a) if rank $\Delta = s_1$ and rank $\Psi = r_1$, by sign Ψ and the (r_1) -pseudonorm $\|\Psi\|$,
 - (b) if rank $\Delta = s_1$ and rank $\Psi < r_1$, by sign Ψ , and
 - (c) if rank $\Delta < s_1$, by sign $P_1 \Psi$ and rank $P_0 \Psi P_0$.

We shall not calculate explicitly the stabilizers, except in case $s_1 = 1$, sufficient for $n \leq 6$.

If $s_1 = 1$, the only cases for sign Δ are $+$, $-$, and 0 . Sign $P_1 \Psi$ is no longer relevant since all columns of Ψ are parallel to one of them and rank Ψ is either 0 or 1 . In the last case, further decompositions are easily done until a zero rank for some ψ_i is obtained, in which case the stabilizer is reductive. We thus obtain three series of stabilizers.

Orbit		Stabilizer	
Sign Δ	Successive rank Ψ		
$+$	$\left\{ \begin{array}{l} 1, 1, \dots, 1, 0 \\ (k-1) \text{ times} \\ (1 \leq k \leq n-3) \end{array} \right\}$	$S(U^{k+1,1}(1, 0) \times GL(n-k-2))$	(4)
0		$S(GL^{k+1,1}(1) \times GL(n-k-2))$	(5)
$-$		$S(U^{k+1,1}(0, 1) \times GL(n-k-2))$	(6)

If the successive decompositions give no zero rank for any Ψ_i , we obtain the following terms, which eventually depend on an additional positive parameter $\|\Psi\|$.

Orbit			Stabilizer	
Sign Δ	Rank Ψ	Norm Ψ		
$+$	$\left\{ \begin{array}{l} 1, 1, \dots, 1 \\ n-3 \end{array} \right\}$	$M \in \mathbb{R}^+$	$S(U^{n-1,1}(1, 0))$	$\sim \mathbb{Z}^n$
0		0	$S(GL^{n-1,1}(1))$	
$-$		$M \in \mathbb{R}^+$	$S(U^{n-1,1}(0, 1))$	

Remark: Analogous series may occur after some steps of decomposition of type Sec. III.A.

IV. DECOMPOSITION IN NILPOTENT CASES

Here, we consider one case where there is "effective nilpotency." This is a starting point for more general nilpotent cases.

Let the first decomposition be $(p+1, 1, 1)$ or $(1, p+1, 1)$ and the second one $s_1 = 1$. We thus obtain G'' with the group law

$$(t, \zeta, w)(t', \zeta', w') = (t' + t - \frac{1}{2}i(\zeta^* w \zeta' e^{-i\theta} - \zeta'^* w' \zeta e^{i\theta}), \zeta + w \zeta' e^{-i\theta}, ww'),$$

where

$$t, t' \in \mathbb{R}, \quad \zeta', \zeta \in \mathbb{C}^p, \quad w, w' \in U(p), \quad \det w = e^{-2i\theta}.$$

If we put $u = we^{-i\theta}$, we observe that

$$w = u \cdot (\det u)^{-1/n} = u \cdot e^{i\theta},$$

i.e., G'' is the n -fold covering of the group Γ_p defined by $(t', \zeta', u')(t, \zeta, u)$

$$= (t + t' - \frac{1}{2}i(\zeta'^* u' \zeta - \zeta^* u' \zeta'), \zeta' + u' \zeta, u'u).$$

Since G'' and Γ_p are locally isomorphic, we study the representations of Γ_p . Since G'' is the central extension $\mathbb{Z}_n \rightarrow G'' \rightarrow \Gamma_p$, all the representations of G'' are ray representations of Γ_p . They are all obtained by allowing the character of the center $U(1)$ of the reductive subgroup $U(p)$ of Γ_p to be any integral multiple of $2\pi/n$. Thus, there correspond n representations of G'' to any representation of Γ_p .

After observing that unfaithful representations of Γ_p are either those of $U(p)$ or those of $U(p) \cdot \mathbb{C}^p$, which are known, we study the faithful representations that occur in the regular representation of Γ_p .

We proceed to the decomposition of the right regular representation. This immediately separates the representations which take different values on the center:

$$(t, \zeta, u): F(t', \zeta', u') \\ \rightarrow F(t + t' + \frac{1}{2}i(\zeta^* u'^* \zeta' - \zeta'^* u' \zeta), \zeta' + u' \zeta, u' u).$$

By the Fourier transform on $\mathbb{C}^p \otimes \mathbb{R} \approx \mathbb{R}(2p+1)$, we obtain

$$(t, \zeta, u): \hat{F}(\lambda, z, u') \rightarrow \exp[i\lambda t + \frac{1}{2}i(\zeta^* z + z^* \zeta)] \\ \times \hat{F}(\lambda, u^{-1}(z - i\lambda \zeta), u' u).$$

Different λ 's give inequivalent representations. Therefore, we now consider $\lambda \neq 0$ to be a constant. We write $\hat{F}(\lambda, z, u') = \exp(-\frac{1}{2}z^* z |\lambda|^{-1}) \cdot \Phi_\lambda(z, u')$; then we introduce a polynomial basis on the z 's for Φ_λ , which spans a dense subspace of $L^2(\mathbb{C}^p)$.

Thus, we have

$$\Phi_\lambda(z, u') \rightarrow \exp[-\frac{1}{2}|\lambda| \zeta^* \zeta + \frac{1}{2}i(z^* \zeta(1 + \epsilon) \\ + \zeta^* z(1 - \epsilon))] \cdot \Phi_\lambda(u^{-1}(z - i\lambda \zeta), u' u),$$

where $\epsilon = \lambda^{-1} \cdot |\lambda|$.

Now, to find a convenient polynomial basis, we search the expression of the infinitesimal generators of the nilpotent part. We take

$$\left. \begin{aligned} Y_\mu &= \frac{\partial}{\partial \bar{\zeta}_\mu} = \frac{\partial}{\partial \zeta_\mu} + \frac{\partial}{\partial \bar{\zeta}_\mu}, \\ Y_\mu &= \frac{\partial}{\partial \eta_\mu} = i \left(\frac{\partial}{\partial \zeta_\mu} - \frac{\partial}{\partial \bar{\zeta}_\mu} \right) \end{aligned} \right\} \text{ or } \left\{ \begin{aligned} 2Z_\mu &= X_\mu + iY_\mu, \\ 2\bar{Z}_\mu &= X_\mu - iY_\mu, \end{aligned} \right.$$

and obtain, for $\lambda > 0$,

$$Z_\mu \Phi_\lambda = i\lambda \frac{\partial \Phi_\lambda}{\partial \bar{z}_\mu}, \quad \bar{Z}_\mu \Phi_\lambda = -i\lambda \frac{\partial \Phi_\lambda}{\partial z_\mu} + i\bar{z}_\mu \Phi_\lambda,$$

and, for $\lambda < 0$,

$$Z_\mu \Phi_\lambda = i\lambda \frac{\partial \Phi_\lambda}{\partial \bar{z}_\mu} + i\bar{z}_\mu \Phi_\lambda, \quad \bar{Z}_\mu \Phi_\lambda = -i\lambda \frac{\partial \Phi_\lambda}{\partial z_\mu}.$$

We now introduce the polynomial basis

$$P_{h,k}^{+|\lambda|}(z, z^*) = (h! k!)^{-1} \left(\bar{z} - |\lambda| \frac{\partial}{\partial \bar{z}} \right)^h z^k,$$

$$P_{h,k}^{-|\lambda|}(z, z^*) = (h! k!)^{-1} \left(z - |\lambda| \frac{\partial}{\partial z} \right)^h \bar{z}^k,$$

where h and k are integral p -tuples. We recall the symbolic notation

$$|k| = |(a, b, c, \dots)| = a + b + c + \dots,$$

$$k! = (a, b, c, \dots)! = a! b! c! \dots,$$

$$z^k = (z_1, z_2, z_3, \dots)^{(a,b,c,\dots)} = z_1^a \cdot z_2^b \cdot z_3^c \dots,$$

$$k \leq k' \Leftrightarrow a \leq a' \text{ and } b \leq b' \text{ and } c \leq c' \text{ and } \dots,$$

$$k - k' = (a - a', b - b', c - c', \dots).$$

We can now write

$$\Phi_{\pm|\lambda|}(z, u) = \sum_{h,k} \varphi_{h,k}(u) \cdot P_{h,k}^{\pm|\lambda|}(z, z^*),$$

where h and k take any value.

We calculate next the restriction of the regular representation to the subgroup $\mathbb{C}^p = \{(0, \zeta, 1)\}$. Let $\rho = |\lambda|$ and $P = P^{-\rho}$ (the case $P^{+\rho}$ can be automatically deduced by interverting z and $\bar{z}$). We can then write

$$P_{h,k} = \sum_m [m! (k-m)! (h-m)!] (-\rho)^{|m|} z^{h-m} \bar{z}^{k-m},$$

where $0 \leq m \leq h$ and $0 \leq m \leq k$.

Carrying out the calculations, we obtain

$$P_{h,k} \rightarrow \sum_{j,m} [(j-m)! m! (h-m)!]^{-1} \\ \times j! (i\bar{\zeta})^{j-m} (i\rho\zeta)^{h-m} \cdot P_{j,k},$$

with

$$0 \leq j \leq \infty, \quad 0 \leq m \leq h, \quad 0 \leq m \leq j.$$

That is,

$$P_{h,k}^{-|\lambda|} \rightarrow \exp(\frac{1}{2}\lambda\zeta^*\zeta) \sum_{0 \leq j \leq \infty} \left(+i\bar{\zeta} + \frac{\partial}{\partial(-i\lambda\zeta)} \right)^j \\ \cdot (-i\lambda\zeta)^h \cdot (h!)^{-1} \cdot P_{j,k}^{-|\lambda|},$$

$$P_{h,k}^{+|\lambda|} \rightarrow \exp(-\frac{1}{2}\lambda\zeta^*\zeta) \sum_{0 \leq j \leq \infty} \left(-i\zeta + \frac{\partial}{\partial(+i\lambda\zeta)} \right)^j \\ \cdot (i\lambda\bar{\zeta})^h \cdot (h!)^{-1} \cdot P_{j,k}^{+|\lambda|}.$$

Thus, the infinitesimal generators of the nilpotent subgroup act only on the first p -tuple. Since this action is independent of the value of k , we may investigate the irreducible subspaces under the whole group by fixing $h = 0$.

We introduce the basis functions

$$\psi_{+|\lambda|}(z, u) = \sum_k \varphi_k(u) P_{0,k}^{+|\lambda|} = \sum_k \varphi_k(u) z^k \cdot (k!)^{-1},$$

$$\psi_{-|\lambda|}(z, u) = \sum_k \varphi_k(u) P_{0,k}^{-|\lambda|} = \sum_k \varphi_k(u) \bar{z}^k \cdot (k!)^{-1}.$$

We consider $\lambda > 0$, i.e., holomorphic functions on $\mathbb{C}^p$, the case $\lambda < 0$ being obtained by changing z to $\bar{z}$ and u to $\bar{u}$.

Decomposing $\varphi_k(u)$ according to the characters of $U(p)$, we have

$$\varphi_k(u) = \sum_{N,h,j,m} D^N(\bar{u})_h^j \cdot e^{im\omega/p} \cdot K_{N,h,j,m}^k,$$

where $e^{i\omega} = \det u$, $\bar{u} \in (SU(p)/Z_p)$, $m \in \mathbb{Z}$, D^N being the UIR of $SU(p)/Z_p$ with maximal weight N .

But the space generated by the $P^{+|\lambda|}$ (respectively, $P^{-|\lambda|}$), $|k|$ being constant, is the space of the UIR of $SU(p)/Z_p$, with maximal weight $(0, 0, \dots, 0, |k|)$ [respectively, $(|k|, 0, \dots, 0)$], the center $U(1)$ being represented by $e^{i\omega} \cdot I \rightarrow e^{-i|k|\omega}$ (respectively, $e^{i|k|\omega}$).

Thus, the problem of decomposing the regular representation into irreducible components is equivalent to a reduction of the tensor product of two irreducible representations of $SU(p)$. Since there is no restriction on the range of N , we have immediately the following result.

Theorem: The regular representation of Γ_p is a direct integral of inequivalent unitary representations, each one labeled by a nonzero real number (character of the center). Furthermore, each of them decomposes into a direct sum of irreducible representations. Equivalence classes of the latter are in 1-to-1 correspondence with UIR's of $U(p)$.

Remark: The theorem holds if we replace Γ_p by $\Gamma_{(p-q) \times q}$ and $U(p)$ by $U(p-q) \times U(q)$. $\Gamma_{(p-q) \times q}$ is defined by exactly the same group law as Γ_p , the reductive group being $U(p-q) \times U(q)$ instead of $U(p)$.

V. RELATION BETWEEN $\mathfrak{g}_n$, $\mathfrak{sp}(n, \mathbb{R})$, AND CREATION-ANNIHILATION OPERATORS

Here we study the connection of the inhomogeneous $\mathfrak{sl}(n, \mathbb{C})$ structure with creation-annihilation operators. In fact, the set of n pairs of these operators forms a nilpotent Lie algebra $\mathfrak{A}_n$ of dimension $2n+1$ with the commutation relations

$$[A_i, A_j^*] = \delta_{ij}I, \quad [A_i, A_j] = [A_i^*, A_j^*] = 0, \\ [A_i, I] = [A_i^*, I] = 0.$$

If now we consider the universal enveloping algebra U_n of $\mathfrak{A}_n$ on the field $\mathbb{R}$ of real numbers and take its quotient by the center I , we find that the set of symmetrized elements of degree two forms itself a Lie algebra, which is isomorphic to $\mathfrak{sp}(n, \mathbb{R})$. One can immediately identify the correspondence between the generators of the two algebras by means of the commutation relations

$$\begin{aligned} & [\tfrac{1}{2}(A_i A_j^* + A_j^* A_i), \tfrac{1}{2}(A_k^* A_h^* + A_h^* A_k^*)] \\ &= \delta_{ik} \tfrac{1}{2}(A_h^* A_j^* + A_j^* A_h^*) + \delta_{ih} \tfrac{1}{2}(A_k^* A_j^* + A_j^* A_k^*), \\ & [\tfrac{1}{2}(A_i A_j^* + A_j^* A_i), \tfrac{1}{2}(A_k A_h + A_h A_k)] \\ &= -\delta_{jk} \tfrac{1}{2}(A_h A_j + A_j A_h) - \delta_{jh} \tfrac{1}{2}(A_i A_k + A_k A_i), \\ & [\tfrac{1}{2}(A_i A_j^* + A_j^* A_i), \tfrac{1}{2}(A_k A_h^* + A_h^* A_k)] \\ &= \delta_{ih} \tfrac{1}{2}(A_k A_j^* + A_j^* A_k) - \delta_{jk} \tfrac{1}{2}(A_i A_h^* + A_h^* A_i), \\ & [\tfrac{1}{2}(A_i A_j + A_j A_i), \tfrac{1}{2}(A_k^* A_h^* + A_h^* A_k^*)] \\ &= \delta_{ik} \tfrac{1}{2}(A_j A_h^* + A_h^* A_j) + \delta_{jk} \tfrac{1}{2}(A_i A_h^* + A_h^* A_i) \\ &+ \delta_{ih} \tfrac{1}{2}(A_j A_k^* + A_k^* A_j) + \delta_{jh} \tfrac{1}{2}(A_i A_k^* + A_k^* A_i). \end{aligned}$$

Remark: To simplify the notations, we do not show the division of every polynomial of degree 2 in A_i and A_j by I , which is, actually, the only correct way to

write commutators in the quotient U_n/I . One cannot drop the division when dealing with representations. The correspondence to the matrix Lie algebra of $\mathfrak{sp}(n, \mathbb{R})$ is

$$\begin{aligned} 2(E_j^i - E_{n+i}^{n+j}) &\sim A_i A_j^* + A_j^* A_i, \\ 2(E_{n+i}^j + E_{n+j}^i) &\sim A_i A_j + A_j A_i, \\ 2(E_i^{n+j} - E_j^{n+i}) &\sim A_i^* A_j^* + A_j^* A_i^*. \end{aligned}$$

On the other hand, since we can always express $\mathfrak{sl}(n, \mathbb{C}) \cdot \mathbb{R}(n^2)$ as a subalgebra of $\mathfrak{sl}(2n, \mathbb{R}) \cdot \mathbb{R}(4n^2)$, we can identify $\mathfrak{g}_n$ as a subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$.¹⁹ The generators of this subalgebra are

$$\begin{aligned} 2X_{ij} &= A_i A_j^* + A_j^* A_i + A_{n+i} A_{n+j}^* + A_{n+j}^* A_{n+i} \\ 2Y_{ij} &= A_i A_{n+j}^* + A_{n+j}^* A_i - A_{n+i} A_j^* - A_j^* A_{n+i} \end{aligned} \in \mathfrak{gl}(n, \mathbb{C}),$$

$$\begin{aligned} 2\sigma_{ij} &= A_i A_j + A_j A_i + A_{n+i} A_{n+j} + A_{n+j} A_{n+i} \\ 2i\alpha_{ij} &= A_i A_{n+j} + A_{n+j} A_i - A_{n+i} A_j - A_j A_{n+i} \end{aligned}$$

$\in \mathbb{R}(n^2)$, σ_{ij} symmetric, real,

$i\alpha_{ij}$ skew-symmetric pure imaginary.

To obtain $\mathfrak{sl}(n, \mathbb{C})$ instead of $\mathfrak{gl}(n, \mathbb{C})$, we must impose a trace condition to the generators, that is, X_{ii} and Y_{ii} do not belong to the subalgebra, but $X_{ii} - X_{jj}$ and $Y_{ii} - Y_{jj}$ do. On the other hand, if we want to consider the generalized Weyl Lie algebra $\mathfrak{B}_n$ instead of the generalized Poincaré $\mathfrak{g}_n$ (the former being the semidirect product of the latter by a real generator, the dilatation), all elements X_{ii} (but not Y_{ii}) appear in $\mathfrak{B}_n$. It is easy to derive representations of W_n from those of G_n , since the dilatation affects only the determinant of α and not its signature. So, if $\det \alpha \neq 0$, there will be one orbit for each signature; the little group may have any real number for determinant, so that the pseudonorm conservation of Secs. II and III drops. In fact, any representation of W_n consists of a direct integral over a class of inequivalent representations of G_n that have identical successive signatures and little groups.

The relation between the structure of $\mathfrak{A}_n$ and $\mathfrak{g}_n$ is now established; we want to examine its effects on representation theory.

Let $\mathcal{A}_n$ be the simply connected Lie group with Lie algebra $\mathfrak{A}_n$. According to the general theory of representations of nilpotent Lie groups,²⁰ all its representations are known. Applying the Mackey-Bruhat method, we see that they form two series. The first series is isomorphic to the set of all representations of the Abelian group $\mathbb{R}(2n)$, the character of the center being trivial. These representations are not faithful.

All faithful representations occur in the second series and are determined by the nontrivial characters

of the center. Passing to the Lie algebra, we see that every representation is characterized by a nonzero, pure imaginary number $i\lambda$, which is the value of the Casimir I . Acting on $L^2(\mathbb{R}^n; dx)$, it may be put into one of two forms

$$\begin{aligned} U_\lambda(A_k) &= \lambda \frac{\partial}{\partial x_k}, & \bar{U}_\lambda(A_k) &= \lambda \frac{\partial}{\partial x_k}, \\ U_\lambda(A_k^*) &= ix_k, & \text{or } \bar{U}_\lambda(A_k^*) &= -ix_k, \text{ with } \lambda > 0, \\ U_\lambda(I) &= i\lambda, & \bar{U}_\lambda(I) &= -i\lambda. \end{aligned}$$

Available results for $\mathfrak{sp}(n; \mathbb{R})$ are obtained by this series only. From U_λ , one obtains

$$2M_{kj} = A_j A_k^* + A_k^* A_j \rightarrow 2x_k \frac{\partial}{\partial x_j} + \delta_{jk},$$

$$2N_{kj} = A_j A_k + A_k A_j \rightarrow i\lambda \frac{\partial^2}{\partial x_j \partial x_k},$$

$$-2S_{kj} = A_j^* A_k^* + A_k^* A_j^* \rightarrow i\lambda^{-1} \cdot x_j x_k.$$

Putting $H_i = M_{ii}$ and $Y_i = \frac{1}{2}\Sigma_{ii}$, $X_i = \frac{1}{2}S_{ii}$ and calculating the value of the first invariant Casimir operator of $\mathfrak{sp}(n, \mathbb{R})$, we find

$$\begin{aligned} C &= \sum_i H_i^2 + (X_i + Y_i)^2 - (X_i - Y_i)^2 \\ &\quad + \sum_{j \neq i} M_{jk} M_{ki} + M_{kj} M_{ji} + S_{jk} \Sigma_{kj} + \Sigma_{kj} S_{jk} \\ &= -\frac{1}{4}(2n^2 + 1). \end{aligned}$$

It is easy to verify that the generators of $\mathfrak{sp}(n, \mathbb{R})$ are expressed as skew-Hermitian operators. On the other hand, representations corresponding to different λ 's are either unitarily equivalent or contragredient, as we see by transforming the expressions above by the operator $f(x) \rightarrow \lambda^{-1/2} f(\lambda^{-1/2} x)$.

Two contragredient representations are not equivalent. Finally, as the example for $n = 1$ shows, these representations are not irreducible, but split into two irreducible inequivalent components acting respectively on the Hilbert spaces $\mathcal{H}^+$ and $\mathcal{H}^-$. $\mathcal{H}^+$ (respectively, $\mathcal{H}^-$) is the closure of the space spanned by polynomials of even (respectively, odd) degree in n real variables, every one multiplied by the function $\exp(-\frac{1}{4}\Sigma x_i^2)$. Thus, we have the following statement.

Proposition 2: A necessary condition for a skew-Hermitian (Schur) irreducible representation of $\mathfrak{sp}(n, \mathbb{R})$ to appear as a direct summand in a "squared" UIR of $\mathcal{A}_n$ is that the Beltrami-Laplace operator C has the value $-\frac{1}{4}(n^2 + \frac{1}{2})$. In this case, the contragredient representation appears also as a direct summand.

An illustration of this result, which establishes that correspondence of structures need not be reflected by a correspondence in representations, is the case $n = 1$.

For $U_\lambda(\mathfrak{U}_1)$, we have

$$Af(x) = \lambda f'(x),$$

$$A^*f(x) = ix f(x),$$

$$If(x) = i\lambda f(x),$$

which gives, for $\mathfrak{sp}(1, \mathbb{R}) \sim \mathfrak{sl}(2, \mathbb{R})$,

$$Hf(x) = xf'(x) + \frac{1}{2}f(x),$$

$$Yf(x) = -\frac{1}{2}if''(x),$$

$$Xf(x) = -\frac{1}{2}ix^2 f(x),$$

$$C = H^2 + 2(XY + YX) = -\frac{3}{4}.$$

Let us call this representation dD and its contragredient $d\bar{D}$. Examining it rapidly, we see that:

(a) The vectors $H^k \cdot \exp(-\frac{1}{2}x^2)$, $X^k \cdot \exp(-\frac{1}{2}x^2)$, and $Y^k \cdot \exp(-\frac{1}{2}x^2)$ (respectively, $H^k \cdot [x \cdot \exp(-\frac{1}{2}x^2)]$, $X^k \cdot [x \cdot \exp(-\frac{1}{2}x^2)]$, and $Y^k \cdot [x \cdot \exp(-\frac{1}{2}x^2)]$) span the same dense subspace of the space $\mathcal{H}^+$ (respectively, $\mathcal{H}^-$), consisting of all even (respectively, odd) square-integrable functions of $\mathbb{R}$ into $\mathbb{C}$.

(b) The compact generator $Y - X = \frac{1}{2}i(x^2 - d^2/dx^2)$ is the energy operator for the 1-dimensional harmonic oscillator. Its spectrum is $(n + \frac{1}{2})$, n being odd for odd eigenfunctions and even for even eigenfunctions. Its period is 4π , while the unity element of the group $\mathfrak{sl}(2, \mathbb{R})$ is equal to $\exp[2\pi(Y - X)]$.

From (a) and (b), we can conclude that

Proposition 3: D is integrable to the direct sum of two inequivalent UIR's of the group $[\mathbb{Z}_2 \cdot SL(2, \mathbb{R})]$,²¹ the twofold covering of $SL(2, \mathbb{R})$, having center $\mathbb{Z}_4$. The expression of the representation D in $\mathcal{H}^+ \oplus \mathcal{H}^-$ is

$$\exp tH : f(x) \rightarrow e^{\frac{1}{2}t} f(e^t x),$$

$$\exp tX : f(x) \rightarrow \exp(-\frac{1}{2}itx^2) f(x),$$

$$\exp tY : f(x) \rightarrow \mathcal{F}^{-1}[\exp(-\frac{1}{2}itx^2) \mathcal{F}f](x).$$

We remark that the integrability and the connection with the harmonic oscillator can be easily generalized for n dimensions.

Coming back to the general case, we notice that representations of $\mathfrak{g}_n$ provide us with the spectra of many generators of $\mathfrak{sp}(2n, \mathbb{R})$. But, since we cannot, in general, take the "square root" of a representation of $\mathfrak{sp}(2n, \mathbb{R})$, we are not able to determine the spectra of the skew-adjoint operators A_i and A_i^* . In particular, the operator $\sum A_i A_i^* + A_i^* A_i$ is contained in $\mathfrak{B}_n$ and its spectrum can be easily calculated for a given representation. In fact, in most interesting cases, its eigenvectors are homogeneous polynomials on the variables (since its form is $\sum x_i \cdot \partial/\partial x_i$), which could be expected, since we identify it with the generalized dilatation.

CONCLUSION

The main results can be summarized in the following way:

(1) A "geometrical" study of involutive automorphisms and antiautomorphisms of the groups G_n is done, thus allowing the formal definition of TCP operators on groups containing the Poincaré group. One must remark that the method used is not specific to the family G_n .

(2) Many series of representations of G_n are found, among them all those which enter the Plancherel formula. However, this formula is not explicitly given, because the Plancherel measure is not yet known for $SU(p, q)$ in general.

(3) The groups $\Gamma_{p \times q} = R \cdot N$ (with R compact-reductive, N nilpotent) are examined, and their regular representation is decomposed into irreducible parts. It seems that the method applied here, which makes use of the properties of a "good" polynomial basis on the infinitesimal generators of N , may also be successfully applied in cases of groups $\Gamma_{p, q}$ (R noncompact) and even more general cases of such semidirect products.

(4) Structural and spectral relations are established between creation-annihilation operators and Lie algebras of the family G_n . Such a treatment has the physical meaning of connecting the group-theoretical and the quantum field-theoretical points of view.

(5) A thorough physical application of these results has still to be done, in order to obtain useful information for group-theoretical problems in domains such as strong interactions or nuclear spectroscopy. We shall deal with these applications in forthcoming papers.

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APPENDIX

We give the list of inequivalent classes of representations for small n in terms of orbits and little groups. This list is complete. Representations are characterized by an orbit decomposition; in this way, two representations which are induced by inequivalent representations of the same little group and which belong to the same orbit are not separated by this classification. [For example, for $n = 2$, the "physical" representations of $\mathcal{P}$ —that is, with strictly positive mass—which have mass M are classified in the same family regardless of their spin. The spin would appear only if one gave the list of all representations of the little group $SU(2)$].

Thus, the description ends at a stabilizer, the representations of which are supposed known. Apart from pseudounitary and linear groups and their direct products, groups like $\Gamma_{(p, q)}$ appear. $\Gamma_{(p, q)}$ is defined in the same manner as $\Gamma_p = \Gamma_{(p, 0)} = \Gamma_{(0, p)}$, but its faithful representations are not yet calculated (except for $\Gamma_{(p, 0)}$ described in Sec. IV).

Orbits may be characterized by a continuous number, $M > 0$. The significance of M is not specified in each case: It may denote the modulus of determinant or a pseudonorm, as it is described in Secs. II and III.

Orbits are always characterized by discrete quantities, like successive signatures and ranks of matrices. These are sometimes described by plus and minus signs or by integers denoting rank, but they are also described by the little group expressed adequately. In the next column, a simpler expression of the stabilizer is given [the notation $GL(k)$ is used for $GL(k, C)$]. The notation $A \wedge B$, where A and B are Abelian groups, denotes a central extension of A by B .

 TABLE I. $n = 2$.

Series	Continuous character	Discrete quantity	Little group	Little group
(2)	M	$\begin{cases} ++ \\ +- \\ -- \end{cases}$	$SU(2, 0)$ $SU(1, 1)$ $SU(0, 2)$	$\sim SU(2)^a$ $\sim SU(1, 1)^b$ $\sim SU(2)^c$
(2)	M	$\begin{cases} + & + \\ - & - \end{cases}$	$S(U(1, 0)^2)$ $S(U(0, 1)^2)$	$\sim \mathbb{Z}_2^4$
(1)		$\begin{cases} + & 0 \\ - & 0 \end{cases}$	$S(U(1, 0) \times GL(1))$ $S(U(0, 1) \times GL(1))$	$\sim U(1)^e$
(1)		0	$SL(2)$	$\sim SL(2)$

^a Positive mass $\sqrt{M}$.

^b Imaginary mass $\sqrt{-M}$.

^c Negative mass $-\sqrt{M}$.

^d Zero mass, continuous spin M .

^e Zero mass, discrete spin.

TABLE II. $n = 3$.

Series	Continuous character	Discrete quantities	Discrete quantities (l.g.)	Little group
(2)	M	$\begin{cases} +++ \\ ++- \\ +-+ \\ --- \end{cases}$	$\begin{cases} SU(3, 0) \\ SU(2, 1) \\ SU(1, 2) \\ SU(0, 3) \end{cases}$	$\begin{cases} SU(3) \\ SU(2, 1) \end{cases}$
		$\begin{cases} ++ & + \\ +- & - \\ -+ & + \\ -- & - \end{cases}$	$\begin{cases} S(U(1, 0) \times U^2(1, 0)) \\ S(U(1, 0) \times U^2(0, 1)) \\ S(U(0, 1) \times U^2(1, 0)) \\ S(U(0, 1) \times U^2(0, 1)) \end{cases}$	$U(1)$
(1)		$\begin{cases} + & + & + \\ - & - & - \end{cases}$	$\begin{cases} S(U^3(1, 0)) \\ S(U^3(0, 1)) \end{cases}$	Z_3
		$\begin{cases} 0 \\ + & 0 \\ - & 0 \end{cases}$	$\begin{cases} SL(3) \\ S(U(1, 0) \times GL(2)) \\ S(U(0, 1) \times GL(2)) \end{cases}$	$\begin{cases} SL(3) \\ U(1) \cdot SL(2) \end{cases}$
(1)		$\begin{cases} ++ & 0 \\ +- & 0 \\ -- & 0 \end{cases}$	$\begin{cases} S(U(2, 0) \times GL(1)) \\ S(U(1, 1) \times GL(1)) \\ S(U(2, 0) \times GL(1)) \end{cases}$	$\begin{cases} U(2) \\ U(1, 1) \end{cases}$
		$\begin{cases} + & + & 0 \\ - & - & 0 \end{cases}$	$\begin{cases} S(U^2(1, 0) \times GL(1)) \\ S(U^2(0, 1) \times GL(1)) \end{cases}$	$U(1)$
(3)		$+ - \quad 1^0$	$S(GL^{2,1}(1)) \cdot R$	$Z_3 \times R$

TABLE III. $n = 4$.

Series	Continuous character	Discrete quantities	Discrete quantities (little group)	Little group
(2)	M	$(p, q, 0)$	$SU(p, q)$	$SU(p, q) \quad p + q = 4$
(2)	M	$(p, q, 1) \quad 1^+$	$S(U(p-1, q) \times U^2(1, 0))$	$\left. \begin{matrix} S(U(p-1, q) \times U(1)) \\ S(U(q, p-1) \times U(1)) \end{matrix} \right\} p + q = 3$
(2)	M	$(p, q, 1) \quad 1^-$	$S(U(p, q-1) \times U^2(0, 1))$	
(1)		$(p, q, 1) \quad 0$	$S(U(p, q) \times GL(1))$	$U(p, q)$
(2)	M	$(p, q, 2)$	$S(U^2(p, q))$	$\left. \begin{matrix} Z_2 \times SU(p, q) \\ U(1) \\ U(1) \\ U(1) \times U(1) \\ U(1) \cdot (SU(p, q) \times SL(2)) \end{matrix} \right\} p + q = 2$
(2)	M		$S(U(p-1, q) \times U^2(1, 0))$	
(2)	M		$S(U(p, q-1) \times U^2(0, 1))$	
(1)			$S(U(p-1, q) \times U^2(1, 0) \times GL(1))$	
(1)			$S(U(p, q-1) \times U^2(0, 1) \times GL(1))$	
(1)			$S(U(p, q) \times GL(2))$	
(2)	M	$(p, q, 3)$	$S(U^4(p, q))$	$\left. \begin{matrix} Z_1 \\ U(1) \\ U(1) \times SL(2) \\ U(1) \cdot SL(3) \end{matrix} \right\} p + q = 1$
(1)			$S(U^3(p, q) \times GL(1))$	
(1)			$S(U^2(p, q) \times GL(2))$	
(1)			$S(U(p, q) \times GL(3))$	
(1)		$(0, 0, 4)$	$SL(4)$	$SL(4)$
—		$(p+1, q+1, 1) \quad 1^0$	$S(GL^{2,1}(1) \times U(p, q)) \cdot (C \wedge R)$	$Z_4 \cdot \Gamma_1 \quad p + q = 1$
		$(1, 1, 2) \quad 1^0$	$S(GL^{2,1}(1) \times GL(1)) \cdot (C \times R) \text{ gives:}$	
(4)		$(1, 1, 2) \quad 1^0$	$S(U^{2,1}(1, 0) \times GL(1))$	$U(1)$
(5)			$S(GL^{2,1}(1) \times GL(1))$	$GL(1)$
(6)			$S(U^{2,1}(0, 1) \times GL(1))$	$U(1)$
(4')	M	$(1, 1, 2) \quad 1^0$	$S(U^{3,1}(1, 0))$	Z_4
(5')			$S(GL^{3,1}(1))$	
(6')	M		$S(U^{3,1}(0, 1))$	

TABLE IV. $n = 6$.

Series	Continuous character	Discrete quantities (little group)	Little group	
(2)	M	$S(U(p, q))$	$S(U(p, q))$	$p + q = 6$
(2)	M	$S(U(p-1, q) \times U^2(1, 0))$	$Z_2 \cdot U(p-1, q)$	$p + q = 5$
(2)	M	$S(U(p, q-1) \times U^2(0, 1))$	$Z_2 \cdot U(p, q-1)$	
(1)		$S(U(p, q) \times GL(1))$	$U(p, q)$	$p + q = 5$
(2)	M	$S(U(p', q') \times U^2(p_1, q_1))$	$S(U(p', q') \times U(p_1, q_1))$	$\begin{cases} p_1 + q_1 = 2 \\ p' + q' = 2 \end{cases}$
(2)	M	$S(U(p', q') \times U^2(p_1, q_1))$	$S(U(p', q') \times U(1))$	$p' + q' = 3$
(1)		$S(U(p', q') \times U^2(p_1, q_1) \times GL(1))$	$U(p', q') \times U(1)$	$p_1 + q_1 = 1$
(1)		$S(U(p, q) \times GL(2))$	$U(1) \cdot (SU(p, q) \times SL(2))$	$p + q = 4$
(2)	M	$S(U^2(p, q))$	$Z_2 \times U(p, q)$	$p + q = 3$
(2)	M	$S(U(p', q') \times U^2(p'', q'') \times U^2(p_2, q_2))$	$U(1) \times U(1)$	$\begin{cases} p_2 + q_2 \\ = p'' + q'' \\ = 1 \end{cases}$
(1)		$S(U(p', q') \times U^2(p_1, q_1) \times GL(1))$	$U(1) \times U(p_1, q_1)$	$p' + q' = 1$
(1)		$S(U(p', q') \times U^2(p_1, q_1) \times GL(2))$	$U(1) \times U(1)$	$p_1 + q_1 = 2$
(1)		$S(U(p, q) \times GL(3))$	$Z_2 \cdot U(p', q')$	
(2)	M	$S(U(p', q') \times U^3(p_1, q_1))$	$U(p', q') \times U(1)$	$p' + q' = 2$
(2)	M	$S(U^2(p', q') \times U^3(p_1, q_1))$	$Z_2 \cdot [U(p', q') \times U(1) \cdot SL(2)]$	$p_1 + q_1 = 1$
(1)		$S(U^2(p', q') \times U^3(p_1, q_1) \times GL(1))$	$U(1) \cdot [SU(p, q) \times SL(3)]$	$p + q = 3$
(1)		$S(U^2(p, q) \times GL(2))$	$Z_2 \times U(p, q)$	$p + q = 2$
(2)	M	$S(U^2(p', q') \times U^4(p_1, q_1))$	$U(1) \times Z_2$	$p' + q' = 1$
(2)	M	$S(U^2(p', q') \times U^4(p_1, q_1) \times GL(1))$	$U(1) \times U(1)$	$p_1 + q_1 = 1$
(1)		$S(U^2(p, q) \times GL(2))$	$U(p, q) \times SL(2)$	$p + q = 2$
(2)	M	$S(U(p', q') \times U^2(p_1, q_1))$	$U(1)$	
(1)		$S(U(p', q') \times U^2(p_1 + q_1) \times GL(1))$	$U(1) \times U(1)$	
(1)		$S(U(p', q') \times U^2(p_1 + q_1) \times GL(2))$	$U(1) \times [U(1) \cdot SL(2)]$	
(1)		$S(U(p', q') \times U^2(p_1 + q_1) \times GL(3))$	$U(1) \times [U(1) \cdot SL(3)]$	
(1)		$S(U(p, q) \times GL(4))$	$U(1) \cdot [SU(p, q) \times SL(4)]$	$p_1 + q_1 = p' + q' = 1$
(2)	M	$S(U^2(p, q))$	Z_6	$p + q = 2$
(1)		$S(U^2(p, q) \times GL(1))$	$U(1)$	
(1)		$S(U^2(p, q) \times GL(2))$	$U(1) \times SL(2)$	
(1)		$S(U^2(p, q) \times GL(3))$	$U(1) \times SL(3)$	$p + q = 1$
(1)		$S(U^2(p, q) \times GL(4))$	$Z_2 \cdot [U(1) \cdot SL(4)]$	
(1)		$S(U(p, q) \times GL(5))$	$U(1) \cdot SL(5)$	
(1)			$SL(6)$	

TABLE IV(bis). $n = 6$.

Series	Continuous character	Discrete quantities	Discrete quantities (little group)	Little group
—		$(p+1, q+1, 1)$	$S(GL^{2+1}(1) \times U(p, q)) \cdot (C^2 \wedge R)$	$Z_6 \cdot \Gamma_{(p, q)}$
—		$(p_1 + p' + 1, q_1 + q' + 1, 2)$	$S(GL^{2+1}(1) \times U(p', q') \times U^2(p_1, q_1)) \cdot (C^2 \wedge R)$	$Z_6 \cdot \Gamma_{1 \times 1}$
(3)		$(2, 2, 2)$	$S(GL^{2+1}(2)) \cdot R^4$	$Z_3 \times G_2$
—		$(p+1, q+1, 2)$	$S(GL^{2+1}(1) \times U(p, q) \times GL(1)) \cdot (C^2 \wedge (C \times R))$ gives:	
—	rank 1:		$S(GL^{2+1}(1) \times U(p, q)) \cdot (C^2 \wedge R)$	$Z_6 \cdot \Gamma_{(p, q)}$
—	rank 0:		$S(GL^{2+1}(1) \times U(p, q) \times GL(1)) \cdot (C^2 \wedge R)$	$U(1) \times [R^+ \cdot \Gamma_{(p, q)}]$
—		$(p+1, q+1, 3)$	$S(GL^{2+1}(1) \times U^2(p, q) \times GL(1)) \cdot ((C \times C) \wedge (C \times R))$ gives	$p+q=2$
—	rank 1:		$S(GL^{2+1}(1) \times U^2(p, q)) \times R$	
—	rank 0:		$S(GL^{2+1}(1) \times U^2(p, q) \times GL(1)) \cdot (C \wedge (C \times R))$ gives	$p+q=1$
—			$S(GL^{2+1}(1) \times U^2(p, q)) \cdot (C \wedge R)$	$Z_2 \times \Gamma_1$
—			$S(GL^{2+1}(1) \times U^2(p, q) \times GL(1)) \cdot (C \wedge R)$	$U(1) \cdot (R^+ \cdot \Gamma_1)$
(3)		$(p+1, q+1, 3)$	$S(GL^{2+1}(1) \times U(p, q)) \times R$	$U(1) \times R$
—		$(p+1, q+1, 3)$	$S(GL^{2+1}(1) \times U(p, q) \times GL(2)) \cdot (C \wedge (C \times R))$ gives:	
—	rank 1		$S(GL^{2+1}(1) \times U(p, q) \cdot (C \wedge R))$	$Z_6 \cdot \Gamma_1$
—	rank 0		$S(GL^{2+1}(1) \times U(p, q) \times GL(1)) \cdot (C \wedge R)$	$U(1) \times (R^+ \cdot \Gamma_1)$
—			$S(GL^{2+1}(1) \times U(p, q) \times GL(2)) \cdot (C \wedge R)$	$[U(1) \cdot SL(2)] \times [(Z_4 \times R^+) \cdot \Gamma_1]$
(4)		$(1, 1, 4)$	$S(U^{3+2}(1, 0) \times GL(1))$	$U(1)$
(5)		$(1, 1, 4)$	$S(GL^{2+2}(1) \times GL(1))$	$GL(1)$
(6)		$(1, 1, 4)$	$S(U^{3+2}(0, 1) \times GL(1))$	$U(1)$
(4')	M	$(1, 1, 4)$	$S(U^{3+2}(1, 0))$	Z_6
(5')		$(1, 1, 4)$	$S(GL^{2+2}(1))$	
(6')		$(1, 1, 4)$	$S(U^{3+2}(0, 1))$	
(4)			$\left\{ \begin{array}{l} + \\ 0 \end{array} \right\}$	$\left. \begin{array}{l} \\ \\ \end{array} \right\} 1 \leq k \leq 3$
(5)			$S(U^{3+1+1}(1, 0) \times GL(4-k))$	
(6)			$S(GL^{k+1+1}(1) \times GL(4-k))$	
(4')		$(1, 1, 4)$	$S(U^{3+1+1}(0, 1) \times GL(4-k))$	
(5')		1^0	$S(U^{3+1}(1, 0))$	
(6')			$S(GL^{3+1}(1))$	
(6')			$S(U^{3+1}(0, 1))$	

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- ¹⁶ The hypothesis of continuity of automorphisms is needed for the proof, as in Ref. 8; even the pure algebraic method of Ref. 15 makes implicitly use of such a hypothesis when passing from rational to real numbers.
- ¹⁷ These results contain the case $n = 2$, since $SU(2) \sim SU^*(2)$ and $SU(1, 1) \sim SL(2, \mathbb{R})$, while for larger n no parity can be defined by $SU(p, q)$; thus $SU^*(2n)$ appears as a good generalization.
- ¹⁸ That is, as defined in the Dynkin diagrams for Lie algebras, see E. B. Dynkin, Transl. Am. Math. Soc., Ser. 2, 6, No. 3 (1960).
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- ²⁰ L. Pukanszky, *Leçons sur les représentations des groupes* (Dunod, Paris, 1965); see also for results on Lie algebra, J. Dixmier, Représentations irréductibles des alg. de Lie nilpotentes, Anais Acad. Brasil Ciencias.
- ²¹ This group has a quotient by $\mathbb{Z}_2$ isomorphic to $SL(2, \mathbb{R})$, but does not contain $SL(2, \mathbb{R})$ as a subgroup; that is why the representations D_+ , D_- , $\bar{D}_+$, and $\bar{D}_-$ do not appear in the series of $SL(2, \mathbb{R})$ calculated by Bargmann.

